Computable randomness is inherently imprecise

Gert de Cooman and Jasper De Bock

IDLab, Ghent University, Belgium

FACULTY OF ENGINEERING AND ARCHITECTURE

GHENT UNIVERSITY

Single forecast game

Forecaster specifies an interval bound $I=[\underline{p},\overline{p}]$ for the expectation of an unknown outcome X in $\{0,1\}$. $\mathscr{C}$ is the set of all closed intervals I in $[0,1]$.

This interval forecast $I=[\underline{p},\overline{p}]$ is a commitment for Forecaster to adopt $\underline{p}$ as supremum buying price and $\overline{p}$ as infimum selling price for the gamble X.

The second player, Sceptic, can now in a second step take Forecaster up on any (combination) of the following commitments:

(i) for any $p\in[0,1]$ such that $p\leq\underline{p}$, and any $\alpha\geq0$, Forecaster must accept the gamble $\alpha[X-p]$;

(ii) for any $q\in[0,1]$ such that $q\geq\overline{p}$, and any $\beta\geq0$, Forecaster must accept the gamble $\beta[q-X]$.

Finally, the third player, Reality, determines the value x of X in $\{0,1\}$.

Sceptic's uncertain rewards

This leads to an uncertain reward (or capital increase) for Sceptic:

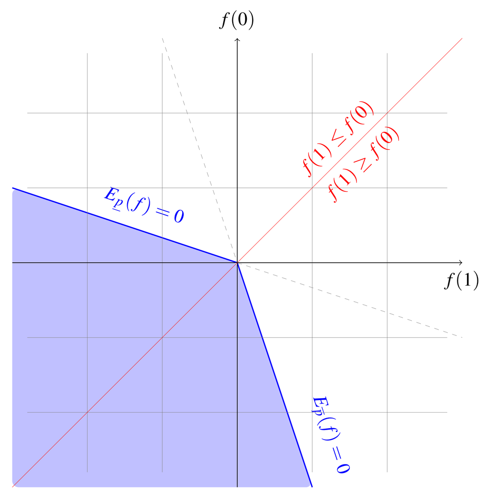
$$\Delta\mathscr{K}=-\alpha[X-p]-\beta[q-X]$$

characterised by

$$\overline{E}_I(\Delta\mathscr{K})\leq0$$

where the upper expectation $\overline{E}_I$ is defined by

$$\overline{E}_I(f)=\max_{p\in I}E_p(f)$$



Forecasting systems

We collect all possible outcome sequences $(x_1,x_2,\dots,x_n,\dots)$ in the set $\Omega:=\{0,1\}^{\mathbb{N}}$. We collect the finite outcome sequences $(x_1,\dots,x_n)$ in the set $\Omega^\lozenge:=\bigcup_{n\in\mathbb{N}_0}\{0,1\}^n$. Finite sequences s in $\Omega^\lozenge$ and infinite sequences ω in Ω are the nodes—called situations—and paths in an event tree with unbounded horizon.

A forecasting system is a map $\gamma\colon\Omega^\lozenge\to\mathscr{C}$, that associates with any situation s in the event tree an interval forecast $\gamma(s)=[\underline{\gamma}(s),\overline{\gamma}(s)]\in\mathscr{C}$. A forecasting system γ is called precise if $\underline{\gamma}=\overline{\gamma}$. Γ denotes the set $\mathscr{C}^{\Omega^\lozenge}$ of all forecasting systems.

Each interval forecast $I_s=\gamma(s)$ corresponds to a local upper expectation $\overline{E}_{I_s}$, with

$$\overline{E}_{\gamma(s)}(f)=\max_{p\in\gamma(s)}E_p(f)=\max_{p\in\gamma(s)}[pf(1)+(1-p)f(0)]$$

so the forecasting system γ turns the event tree into an imprecise probability tree.

Computable randomness

A map $M\colon\Omega^\lozenge\to\mathbb{R}$ is a supermartingale for γ if $\overline{E}_{\gamma(s)}(\Delta M(s))\leq0$ for all $s\in\Omega^\lozenge$. In other words, it is a possible capital process for Sceptic.

We call an event $A\subseteq\Omega$ strictly null if there is some non-negative supermartingale T for γ that converges to $+\infty$ on A, meaning that $\lim_{n\to+\infty}T(\omega^n)=+\infty$ for all $\omega\in A$. The complement A^c of a strictly null event A is never empty. A property that holds on A^c is said to hold strictly almost surely, or for strictly almost all outcome sequences.

An outcome sequence ω is computably random for γ if all computable non-negative supermartingales T for γ remain bounded above on ω, meaning that $\sup_{n\in\mathbb{N}}T(\omega^n)<+\infty$.

$$\Gamma_{\mathrm{C}}(\omega):=\{\gamma\in\Gamma\colon\omega\text{ is computably random for }\gamma\}$$

is the set of all forecasting systems for which ω is computably random.

Proposition 1. *All paths are computably random for the vacuous forecasting system: $\gamma_v\in\Gamma_{\mathrm{C}}(\omega)$ for all $\omega\in\Omega$, so $\Gamma_{\mathrm{C}}(\omega)$ is never empty.*

More conservative (or imprecise) forecasting systems have more computably random sequences.

Proposition 2. *Let ω be computably random for a forecasting system γ. Then ω is also computably random for any forecasting system γ^* such that $\gamma\subseteq\gamma^*$, meaning that $\gamma(s)\subseteq\gamma^*(s)$ for all $s\in\Omega^\lozenge$.*

Theorem 3. *Consider any forecasting system γ. Then strictly almost all outcome sequences are computably random for γ in the imprecise probability tree that corresponds to γ.*

Corollary 4. *For any sequence of interval forecasts $(I_1,\dots,I_n,\dots)$ there is a forecasting system given by $\gamma(x_1,\dots,x_n):=I_{n+1}$ for all $(x_1,\dots,x_n)\in\{0,1\}^n$ and all $n\in\mathbb{N}_0$, and associated imprecise probability tree such that strictly almost all—and therefore definitely at least one—outcome sequences are computably random for γ in the associated imprecise probability tree.*

Constant forecasts

The stationary forecasting system γ_I assigns the same interval forecast I to all nodes:

$$\gamma_I(s):=I\text{ for all }s\in\Omega^\lozenge$$

Consider all interval forecasts for which the corresponding stationary forecasting system makes ω computably random:

$$\mathscr{C}_{\mathrm{C}}(\omega):=\{I\in\mathscr{C}\colon\gamma_I\in\Gamma_{\mathrm{C}}(\omega)\}$$

Proposition 5 (Non-emptiness). *For all $\omega\in\Omega$, $[0,1]\in\mathscr{C}_{\mathrm{C}}(\omega)$, so any sequence of outcomes ω has at least one stationary forecast that makes it computably random: $\mathscr{C}_{\mathrm{C}}(\omega)\neq\emptyset$.*

Proposition 6 (Increasingness). *Consider any $\omega\in\Omega$ and any $I,J\in\mathscr{C}$. If $I\in\mathscr{C}_{\mathrm{C}}(\omega)$ and $I\subseteq J$, then also $J\in\mathscr{C}_{\mathrm{C}}(\omega)$.*

Proposition 7 (Closure). *For any $\omega\in\Omega$ and any two interval forecasts I and J: if $I\in\mathscr{C}_{\mathrm{C}}(\omega)$ and $J\in\mathscr{C}_{\mathrm{C}}(\omega)$ then $I\cap J\neq\emptyset$, and $I\cap J\in\mathscr{C}_{\mathrm{C}}(\omega)$.*

Hence, $\mathscr{C}_{\mathrm{C}}(\omega)$ is a set filter, and

$$\bigcap\mathscr{C}_{\mathrm{C}}(\omega)=:[\underline{p}_{\mathrm{C}}(\omega),\overline{p}_{\mathrm{C}}(\omega)]$$

is a non-empty closed interval. Also

$$[0,1]\cap[\underline{p}_{\mathrm{C}}(\omega)-\epsilon_1,\overline{p}_{\mathrm{C}}(\omega)+\epsilon_2]\in\mathscr{C}_{\mathrm{C}}(\omega)$$

for all $\epsilon_1>0$ and $\epsilon_2>0$.

Examples

As a first example, fix any $p\leq q$ in $[0,1]$, and the forecasting system $\gamma_{p,q}$ with

$$\gamma_{p,q}(x_1,\dots,x_n):=\begin{cases}p&\text{if }n\text{ is odd}\\q&\text{if }n\text{ is even}\end{cases}$$

Proposition 9. *Consider any ω that is computably random for the precise forecasting system $\gamma_{p,q}$. Then for all $I\in\mathscr{C}$, $I\in\mathscr{C}_{\mathrm{C}}(\omega)$ if and only if $[p,q]\subseteq I$.*

Hence, $\underline{p}_{\mathrm{C}}(\omega)=p$ and $\overline{p}_{\mathrm{C}}(\omega)=q$.

As a second example, consider the sequence $\{p_n\}_{n\in\mathbb{N}}$ in $[0,1]$ converging to $1/2$:

$$p_n:=\frac{1}{2}+(-1)^n\delta_n\text{, with}$$

$$\delta_n:=e^{-\frac{1}{n+1}}\sqrt{e^{\frac{1}{n+1}}-1}\text{ for all }n\in\mathbb{N}$$

and the forecasting system $\gamma_{\sim1/2}$ with

$$\gamma_{\sim1/2}(x_1,\dots,x_{n-1}):=p_n\text{ for all }n\in\mathbb{N}$$

Proposition 10. *Consider any ω that is computably random for the precise forecasting system $\gamma_{\sim1/2}$. Then for all $I\in\mathscr{C}$, $I\in\mathscr{C}_{\mathrm{C}}(\omega)$ if and only if $\min I<1/2$ and $\max I>1/2$.*

Hence $\underline{p}_{\mathrm{C}}(\omega)=\overline{p}_{\mathrm{C}}(\omega)=1/2$.

Church randomness

Computable randomness implies an intuitive limiting frequencies result:

Theorem 8 (Church randomness). *Consider any outcome sequence $\omega=(x_1,\dots,x_n,\dots)$ in Ω and any stationary interval forecast $I=[\underline{p},\overline{p}]\in\mathscr{C}_{\mathrm{C}}(\omega)$ that makes ω computably random. Then for any computable selection process $S\colon\Omega^\lozenge\to\{0,1\}$ such that $\sum_{k=0}^{n}S(x_1,\dots,x_k)\to+\infty$:*

$$\underline{p}\leq\liminf_{n\to+\infty}\frac{\sum_{k=0}^{n-1}S(x_1,\dots,x_k)x_{k+1}}{\sum_{k=0}^{n-1}S(x_1,\dots,x_k)}\leq\limsup_{n\to+\infty}\frac{\sum_{k=0}^{n-1}S(x_1,\dots,x_k)x_{k+1}}{\sum_{k=0}^{n-1}S(x_1,\dots,x_k)}\leq\overline{p}$$

$\square$ $\gamma(\square)=I_\square$

1 $\gamma(1)=I_1$

11 $\gamma(11)=I_{11}$

10 $\gamma(10)=I_{10}$

0 $\gamma(0)=I_0$

01 $\gamma(01)=I_{01}$

00 $\gamma(00)=I_{00}$

111 110 101 100 011 010 001 000